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Relations between the Coulomb Gas Picture and Conformal Invariance of Two-Dimensional Critical Models

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Partition functions of critical 2D models on a torus can be derived from their microscopic formulation and their free field representation in the continuum limit. This is worked out explicitly for the $O(n)$ and Q-state Potts model. For n or Q integer we recover results obtained from conformal invariance, but our procedure also extends to nonintegral values. In the latter case the expansion on characters of the Virasoro algebra involves real coefficients of either sign. The operator content of both models is discussed in detail.

KEY WORDS: Coulomb gas; conformal invariance; partition functions.

1. INTRODUCTION

Two-dimensional critical models have been studied recently using two different approaches. On one hand, conformal invariance has proved to be a very strong constraint.[1,2] On the other hand, it is well known[3] that most two-dimensional models renormalize at criticality onto a Gaussian free-field theory (Coulomb gas). This property has been mainly used so far to compute exact critical exponents (see Ref. 4 for a review), but it is probably deeply related to the conformal invariance approach. Indeed, Dotsenko and Fateev[5] have shown that the existence of a nonzero four-point correlation function in a free field theory supplemented by a charge at infinity leads naturally to dimensions given by the Kac formula, and that the introduction of the so-called screening operators allows explicit computations. Also, Nienhuis and Knops[6] have discussed the physical inter-

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pretation and Coulomb gas construction of spinor operators for the Potts model, which were first obtained by conformal invariance considerations.

In this work, we want to make these relations more precise by considering partition functions on a torus. The torus $\mathbb{T}$ is defined by two periods ω_1, ω_2. For models with a transfer matrix one can write[7]

$$Z = \mathrm{Tr}(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}) \tag{1.1}$$

where $\tau = \omega_2/\omega_1 = \tau_R + i\tau_I$ is the modular ratio and $q = \exp(2i\pi\tau)$. Decomposing the trace on the various irreducible representations of the two Virasoro algebras gives

$$Z = \sum_{h,\bar{h}} N_{h\bar{h}} \chi_h \chi_{\bar{h}}^* \tag{1.2}$$

the nonnegative integer $N_{h\bar{h}}$ representing the multiplicity of the operator with dimensions h, $\bar{h}$. The constraint of modular invariance leads then to a classification of families of possible partition functions.[8-10] For the three-state Potts model, for instance, the knowledge of the central charge and a few dimensions allowed Cardy[7] to determine completely Z, and thus the whole operator content of the theory. We show in the following that such expressions for Z can in fact be derived starting from the microscopic definition of the model, and using its mapping onto a Coulomb gas.

In Section 2 we consider the XY and F models. Both have $c = 1$, and can be described by a free field with defect lines on a torus. Their partition functions are thus simply the Coulombic partition functions we introduced in a preceding work[11] (see also Ref. 12).

In Section 3 and 4 we work out in detail the $O(n)$ and Q-state Potts models. Since the corresponding central charges are smaller than 1, the construction of Z implies an additional "floating" electric charge, which is the equivalent of the charge at infinity in Refs. 5 and 13. We recover already known results for n, Q integer. For n, $Q \in \mathbb{R}$, we obtain expressions similar to (1.2), but with real and even negative coefficients, which seems to imply the absence of a transfer matrix formulation. The operator content is discussed in detail; it reproduces in particular thermal and magnetic series as conjectured by Dotsenko and Fateev.[5] Section 5 contains a few final comments.

In the Appendix, we show how to reexpress all the minimal partition functions of Ref. 10 as linear combinations of Coulombic partition functions.

2. FREE FIELD FORMULATION

1. It has been known for a long time[3] that most two-dimensional statistical mechanics models at criticality renormalize onto a Gaussian free field with action

$$\mathscr{A} = \frac{g}{4\pi} \int |\nabla\varphi|^2 \, d^2x \tag{2.1}$$

where g is a coupling constant. In the theory (2.1), the basic operators are the exponentials of the free field $\mathcal{O}_e = \exp(ie\varphi)$, satisfying

$$\langle \mathcal{O}_e(\mathbf{r}) \, \mathcal{O}_{-e}(\mathbf{r}') \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-e^2/g} \tag{2.2}$$

and the dual operators $\mathcal{O}_m$, the correlation functions of which are obtained by imposing a discontinuity of $2m\pi$ on the field φ when one crosses a line connecting $\mathbf{r}$ to $\mathbf{r}'$,

$$\langle \mathcal{O}_m(\mathbf{r}) \, \mathcal{O}_{-m}(\mathbf{r}') \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-gm^2} \tag{2.3}$$

Since (2.2) or (2.3) can be written as exponentials of the Coulomb interaction $\log(|\mathbf{r} - \mathbf{r}'|)$, one refers also to (2.1) as a two-dimensional classical Coulomb gas[3]: e and m are called, respectively, electric and magnetic charges. Combining (2.2) and (2.3), one gets a more general object $\mathcal{O}_{em}$,

$$\langle \mathcal{O}_{em}(\mathbf{r}) \, \mathcal{O}_{-e-m}(\mathbf{r}') \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-e^2/g - gm^2} \exp[-2iem\alpha(\mathbf{r} - \mathbf{r}')] \tag{2.4}$$

where $\alpha(\mathbf{r} - \mathbf{r}')$ is the angle of the $\mathbf{r} - \mathbf{r}'$ vector with an arbitrary direction. Thus $\mathcal{O}_{em}$ has a dimension x and a spin s given by

$$x_{em} = e^2/2g + gm^2/2, \qquad s_{em} = em \tag{2.5}$$

We recall also the existence of a duality transformation[3] for (2.1), which has the effect

$$g \to 4/g, \qquad \mathcal{O}_{em} \to \mathcal{O}_{2m,e/2}, \qquad x, s \text{ unchanged} \tag{2.6}$$

2. So far, the mapping of discrete models onto (2.1) was mainly used to calculate critical exponents. This requires the knowledge of the renormalized coupling constant g and the formulation of the operators one wants to study in terms of $\mathcal{O}_{em}$ (see Ref. 4 for a review). In Ref. 11 we started to use this mapping to determine also the continuum limit of partition functions on a torus, an object much studied recently in the light of conformal invariance.[7-10]

For a free field on a torus $\mathbb{T}$, with the action as in (2.1) integrated over $\mathbb{T}$, a properly renormalized expression for the partition function

$$Z_1(g) = \int_{\varphi \text{ periodic}} [D\varphi]\, e^{-\mathscr{A}} \tag{2.7}$$

is[8]

$$Z_1(g) = \frac{\sqrt{g}}{\tau_I^{1/2} \eta(q)\, \eta(\bar{q})} \tag{2.8}$$

η is the Dedekind function

$$\eta(q) = q^{1/24} \prod_{N=1}^{\infty} (1 - q^N), \qquad q = \exp(2i\pi\tau) \tag{2.9}$$

The dependence on the coupling constant g comes from the existence of a zero mode, the subtraction of which forbids the rescaling of φ.

Z_1 enjoys the important property of modular invariance, i.e., invariance under the modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \tag{2.10}$$

The behavior at small q (cylinder limit)[13,14]

$$Z \sim (q\bar{q})^{-c/24}, \qquad q \to 0 \tag{2.11}$$

is in agreement with the value $c = 1$ for the central charge of the Gaussian free field.

However, most models have a partition function more complicated than (2.8). This is due in part to special boundary conditions in (2.7) which are generated by the mapping onto (2.1). Consider, for instance, the XY model,[15] defined by the action

$$\mathscr{A} = -\frac{1}{T} \sum_{\langle jk \rangle} \mathbf{S}_j \cdot \mathbf{S}_k \tag{2.12}$$

where $\mathbf{S}$ is a two-component unit vector, and the sum is taken over nearest neighbor pairs of a regular lattice. It is known that in the whole low-temperature critical phase $T \leqslant T_c$, vortices remain bound and (2.12) maps onto (2.1), which corresponds to a spin-wave approximation with a renormalized temperature.[16] On a torus, however, vortex lines that are wrapped along noncontractible loops remain in the continuum limit. For a variation of angle equal to $2\pi m$ ($2\pi m'$) along ω_1 (ω_2), the corresponding continuum limit is[11]

$$Z_{m',m}(g)=\int_{\substack{\delta_1\varphi=2\pi m\\ \delta_2\varphi=2\pi m'}}[D\varphi]\,e^{-\mathscr{A}} \tag{2.13}$$

In this limit, the vorticity of **S** is transformed into a discontinuity of the field φ, referred to as "frustration" in the following. Relation (2.13) is readily evaluated[11] using the classical solution (such that $\Delta\varphi=0$)

$$Z_{m',m}(g)=Z_1(g)\exp\left[-\pi g\,\frac{m'^2+m^2(\tau_R^2+\tau_I^2)-2\tau_R mm'}{\tau_I}\right] \tag{2.14}$$

Relation (2.14) is not modular-invariant. One can verify that it transforms in the same way as the frustrations

$$Z_{m',m}\left(\frac{a\tau+b}{c\tau+d}\right)=Z_{am'+bm,cm'+dm}(\tau) \tag{2.15}$$

A simple modular-invariant object is then obtained by summing over m, m', giving what we have called[11] a Coulombic partition function

$$Z_c[g,1]=\sum_{m',m\in\mathbb{Z}}Z_{m',m}(g) \tag{2.16}$$

After a Poisson transformation one finds

$$\begin{aligned} Z_c[g,1]&=\frac{1}{\eta\bar{\eta}}\sum_{e,m\in\mathbb{Z}}q^{(e/\sqrt{g}+m\sqrt{g})^2/4}\bar{q}^{(e/\sqrt{g}-m\sqrt{g})^2/4}\\ &=\frac{1}{\eta\bar{\eta}}\sum q^{\Delta_{em}}\bar{q}^{\bar{\Delta}_{em}} \end{aligned} \tag{2.17}$$

where the conformal weights[1] (in the $c=1$ theory) are given by

$$\begin{aligned} \Delta_{em}+\bar{\Delta}_{em}&=x_{em}=\frac{e^2}{2g}+\frac{g}{2}m^2\\ \Delta_{em}-\bar{\Delta}_{em}&=s_{em}=em \end{aligned} \tag{2.18}$$

For the XY model, the value of the renormalized coupling constant g is known[4] at the Kosterlitz–Thouless (KT) point T_c only, where it takes the value 4. We thus proposed in Ref. 11 that Z_c [4, 1] is the corresponding partition function on a torus. This agrees with the recent work of Yang[12] on the Ashkin–Teller model, a special point of which is the KT point. This approach does not account for logarithmic terms.[15]

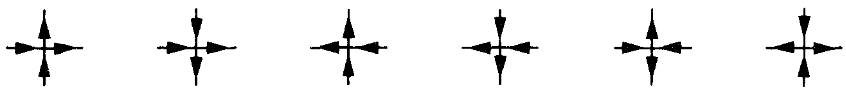
It is worth noticing that the partition function (2.16) describes a bosonic free field considered as an *angle*, i.e., a free field living on a circle. As such, this is the simplest instance of the "toroidal compactification" extensively used in recent work on string theory.[17] Conversely, compactification on a higher dimensional torus yields other modular-invariant functions, which might correspond to interesting statistical mechanical models (with c integer >1).

3. Another example of interest is the F model,[18] defined by putting arrows on the edges of the square lattice with six possible vertex configurations (Fig. 1), the Boltzmann weights of the vertices being $W_1 = \cdots = W_4 = 1$, $W_5 = W_6 = \exp(1/T)$. The whole high-temperature phase $[T > (\log 2)^{-1}]$ is critical and it can be studied by reformulating it as a solid on solid (SOS) surface model.[4] For this purpose one simply introduces height variables φ on the faces of the square lattice such that two neighboring φ differ by $\pm\varphi_0$, the highest being on the left of each arrow. Then this SOS model is argued to renormalize onto (2.1) and the corresponding coupling constant can be evaluated. The standard choice $\varphi_0 = \pi/2$ gives[4]

$$g(T) = \frac{8}{\pi} \sin^{-1} \frac{e^{1/T}}{2} \tag{2.19}$$

With this local definition of the variables φ it is clear that discontinuities may be generated on a torus. Preserving the antiferroelectric symmetry of the model requires to taking an even number of sites around each period, which gives discontinuities of φ multiple of π. This leads us to the expression for the partition function in the continuum limit

$$\mathscr{Z}_F(T) \to 2 \sum_{m',m \in \mathbb{Z}/2} Z_{m',m}(g) \tag{2.20}$$



W_1 W_2 W_3 W_4 W_5 W_6

Fig. 1. The six allowed vertices of the F model.

i.e., after Poisson transformation

$$\mathscr{Z}_F(T) \to Z_c[g, \tfrac{1}{2}] = \frac{1}{\eta\bar{\eta}} \sum_{\substack{e \in 2\mathbb{Z} \\ m \in \mathbb{Z}/2}} q^{\Delta_{em}} \bar{q}^{\bar{\Delta}_{em}} \tag{2.21}$$

The global normalization 2 in (2.20) is introduced simply to give a nondegenerate identity operator.

More generally, we find it convenient to introduce partition functions summed over frustration multiples of $2\pi f$

$$Z_c[g, f] = f \sum_{m', m \in f\mathbb{Z}} Z_{m',m}(g) = \frac{1}{\eta\bar{\eta}} \sum_{\substack{e \in \mathbb{Z}/f \\ m \in f\mathbb{Z}}} q^{\Delta_{em}} \bar{q}^{\bar{\Delta}_{em}} \tag{2.22}$$

It has the symmetries

$$Z_c[g, f] = Z_c[1/g, 1/f] = Z_c[gf^2, 1] \tag{2.23}$$

corresponding to the invariance under (2.6).

4. We notice, however, that expressions like (2.22) are only relevant for models with central charge $c = 1$. To describe situations with $c < 1$ one must introduce a pair of electric charges $\pm e_0$ "at infinity." This is easily done on a plane[5] or on a cylinder,[13] with the corresponding modification

$$c = 1 - 6e_0^2/g \tag{2.24}$$

since the small-q behavior (2.11) is now given by

$$(q\bar{q})^{\Delta_{e_0,O} - 1/24}$$

If one wants to calculate toroidal partition function for $c < 1$ models using their Coulomb gas mapping, it is then necessary to understand how these charges can be introduced on a torus. We shall study this question for the $O(n)$ or Q-state Potts models (n, $Q \in \mathbb{R}$) in the following sections.

3. PARTITION FUNCTIONS OF CRITICAL $O(n)$ ($n \in \mathbb{R}$) MODELS ON A TORUS

1. We consider the $O(n)$ model on the honeycomb lattice[4] defined initially for n integer by

$$\mathscr{Z}_n = \int \prod_i dS_i \prod_{\langle jk \rangle} \left(1 + \frac{1}{T} \mathbf{S}_j \mathbf{S}_k\right) \tag{3.1}$$

where S is an n-component vector such that $|\mathbf{S}|^2 = n$. It can be analytically continued to $n \in \mathbb{R}$ using a high-temperature expansion

$$\mathscr{Z}_n = \sum_{\text{graphs}} \left(\frac{1}{T}\right)^{\mathscr{N}_B} n^{\mathscr{N}_p} \tag{3.2}$$

In (3.2) the graphs are formed by $\mathscr{N}_p$ nonintersecting self-avoiding loops (or "polygons") of total length $\mathscr{N}_B$. The model is known to be critical for $n \in [-2, 2]$. It can be transformed into an SOS model[4] by introducing height variables φ on the centers of the hexagons. An arbitrarily oriented polygon corresponds then to a wall between two regions of constant height, with a step $\pm\varphi_0$, the highest φ being on the left of each arrow. The Boltzmann weight consists of a factor $1/T$ for each bond, times e^{iv} (e^{-iv}) for each left (right) turn. Then, since the difference between the numbers of left and right turns for a polygon on the honeycomb lattice on a plane is $n_l - n_r = \pm 6$, one has $\mathscr{Z}_n = \mathscr{Z}_{\text{SOS}}$ if $n = 2\cos 6v$.

The renormalized coupling constant is then evaluated[4] and for the standard choice $\varphi_0 = \pi$ one gets

$$n = -2\cos \pi g, \qquad g \in [1, 2] \tag{3.3}$$

2. On a torus, however, $\mathscr{Z}_n \neq \mathscr{Z}_{\text{SOS}}$, since polygons that wrap around it have $n_l = n_r$. The above weights thus describe a modified partition function[11]

$$\tilde{\mathscr{Z}}_n = \sum_{\text{graphs}} \left(\frac{1}{T}\right)^{\mathscr{N}_B} n^{\mathscr{N}_p} 2^{\tilde{\mathscr{N}}_p} \tag{3.4}$$

where $\tilde{\mathscr{N}}_p$ is the number of polygons nonhomotopic to a point. In the SOS model, there are also frustrations, exactly as in the F-model case (Section 2), leading then to the continuum limit of (3.4) at criticality

$$\tilde{\mathscr{Z}}_n \to Z_c[g, 1/2] = Z_c[g/4, 1] \tag{3.5}$$

If $n = 2$, $\tilde{\mathscr{Z}}_2 = \mathscr{Z}_2$. With the previous conventions (3.3), $g = 1$, we thus find $\mathscr{Z}_{XY} \to Z_c[1, 1/2] = Z_c[4, 1]$, in agreement with the above results (Section 2).

In the limit $q \to 0$, the leading contribution to $\mathscr{Z}_n$ comes from configurations with polygons wrapping around the axis of the cylinder. It is easy to give them the weight (3.2) by putting charges $\pm e_0$ at infinity (i.e., at the two ends of the cylinder) such that $n = 2\cos \pi e_0$. This gives[13]

$$e_0 = \pm(g-1) \bmod 2, \qquad c = 1 - 6(g-1)^2/g \tag{3.6}$$

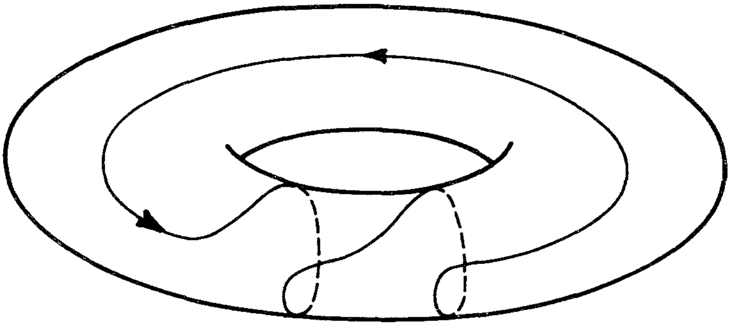


Fig. 2. Example of a polygon $\mathscr{P}$ nonhomotopic to a point. In the natural basis of the torus it defines frustrations $\delta_1 \varphi = 2\pi$, $\delta_2 \varphi = \pi$.

We now discuss how one can get the correct weights on a torus. For this purpose we need two topological properties.

(i) A non-self-intersecting polygon $\mathscr{P}$ nonhomotopic to a point defines two frustrations $\delta_1 \varphi = n_1 \pi$, $\delta_2 \varphi = n_2 \pi$ along two independent periods ω_1, ω_2. Then $|n_1|$ and $|n_2|$ are coprimes. (Fig. 2)

(ii) If two unoriented such polygons $\mathscr{P}$, $\mathscr{P}'$ coexist (i.e., do not intersect) on the torus, then they are homotopic. (Fig. 3)

Property (i) may be proved following the line of argument of Ref. 19. By a suitable modular transformation (2.15), i.e., a change of basis (ω_1, ω_2), the system of frustrations $\{n_1, n_2\}$ may take the form $\{|n_1| \wedge |n_2| \operatorname{sgn}(n_1 n_2), 0\}$ ($a \wedge b$ denotes the greatest common divisor of a and b, and by convention $a \wedge 0 = a$). Connectivity and non-self-intersection of $\mathscr{P}$ then guarantee $|n_1| \wedge |n_2| = 1$. Property (ii) follows, since in the basis that makes $\mathscr{P}$ trivial (frustration $= \{\pm 1, 0\}$), $\mathscr{P}'$ may only be $\{\pm 1, 0\}$.

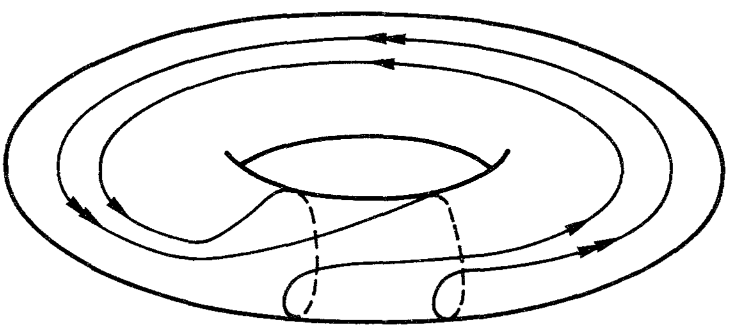


Fig. 3. Two coexisting polygons on the torus. They define the same set of frustrations.

For a given configuration in (3.2), the frustrations then read

$$\delta\varphi_1 = \pi n_1 \sum \varepsilon_i, \qquad \delta\varphi_2 = \pi n_2 \sum \varepsilon_i \tag{3.7}$$

In this expression, the sum is taken over all polygons nonhomotopic to a point. Because of (ii), these are homotopic and thus define the same frustrations $n_1\pi$ and $n_2\pi$, up to a sign depending on their orientation, $\varepsilon_i = \pm 1$. By convention, $\varepsilon_i = 1$ corresponds, say, to $n_1 > 0$, or $n_2 > 0$ if $n_1 = 0$. Since $|n_1|$ and $|n_2|$ are coprimes, one has

$$\sum \varepsilon_i = \pm \frac{|\delta\varphi_1|}{\pi} \wedge \frac{|\delta\varphi_2|}{\pi}$$

the sign depending on the topology of the basic polygon. In any case

$$\cos \pi e_0 \sum \varepsilon_i = \cos\left(\pi e_0 \frac{|\delta\varphi_1|}{\pi} \wedge \frac{|\delta\varphi_2|}{\pi}\right) \tag{3.8}$$

One can then get $\mathscr{Z}_n$ by multiplying the SOS weight by the term (3.8). Since

$$\sum_{\{\varepsilon_i = \pm 1\}} \cos \pi e_0 \sum \varepsilon_i = \prod_i \sum_{\varepsilon_i = \pm 1} e^{i\pi e_0 \varepsilon_i} = n^{\mathcal{N}_p}$$

we find the desired weight for each polygon nonhomotopic to a point. Because of the modular transformations of the frustrations, this procedure does not depend on the choice of ω_1, ω_2. We thus find that the continuum limit of $\mathscr{Z}_n$ at criticality is

$$\mathscr{Z}_n \to \hat{Z}[g, e_0] = \sum_{M', M \in \mathbb{Z}} Z_{M',M}(g/4) \cos(\pi e_0 M' \wedge M) \tag{3.9}$$

which is clearly modular-invariant.

3. We now check (3.9) using some known results. For $n = 2$, $e_0 = 0$, and (3.9) is correct by construction. For $n = 1$, $e_0 = 1/3$. The sum in (3.9) can be decomposed on the different congruence classes of $M' \wedge M \bmod 6$,

$$\hat{Z}\left[g, \frac{1}{3}\right] = \left[\sum_{M' \wedge M = 0 \bmod 6} + \frac{1}{2} \sum_{M' \wedge M = 1,5 \bmod 6} - \frac{1}{2} \sum_{M' \wedge M = 2,4 \bmod 6} - \sum_{M' \wedge M = 3 \bmod 6}\right] Z_{M',M}\left(\frac{g}{4}\right) \tag{3.10}$$

Then one has

$$\begin{aligned}\sum_{M' \wedge M = 0 \bmod 6} Z_{M',M} &= \sum_{M',M \in 6\mathbb{Z}} Z_{M',M} = Z_c[g/4, 6]/6 \\ \sum_{M' \wedge M = 3 \bmod 6} Z_{M',M} &= Z_c[g/4, 3]/3 - Z_c[g/4, 6]/6 \\ \sum_{M' \wedge M = 2,4 \bmod 6} Z_{M',M} &= Z_c[g/4, 2]/2 - Z_c[g/4, 6]/6 \\ \sum_{M' \wedge M = 1,5 \bmod 6} Z_{M',M} &= Z_c[g/4, 1] - Z_c[g/4, 2]/2 \\ &\quad - Z_c[g/4, 3]/3 + Z_c[g/4, 6]/6\end{aligned} \tag{3.11}$$

Thus

$$\hat{Z}[g, 1/3] = (Z_c[g/4, 6] - Z_c[g/4, 3] - Z_c[g/4, 2] + Z_c[g/4, 1])/2 \tag{3.12}$$

for $n=1$, $g=4/3$ and the symmetries (2.23) leave simply

$$\hat{Z}[4/3, 1/3] = (Z_c[4/3, 3] - Z_c[4/3, 1])/2 \tag{3.13}$$

This is exactly the expression for the partition function of the Ising model, as demonstrated in Ref. 11 (see also the Appendix).

For $n=0$, $e_0 = 1/2$, and one finds in a similar way

$$\hat{Z}[g, 1/2] = (Z_c[g, 2] - Z_c[g, 1])/2 \tag{3.14}$$

The relevant value of g is 3/2, and one has

$$\hat{Z}[3/2, 1/2] = (Z_c[3/2, 2] - Z_c[3/2, 1])/2 = 1 \tag{3.15}$$

by Euler's identity. This agrees[11] with the well-known result $\mathscr{Z}_{n=0} = 1$, (3.2).

4. We turn now to the case of an arbitrary value of n. Then, Z may not be expressed as a sum of a finite number of Coulombic partition functions (at least for e_0 irrational). We calculated instead (3.9) term by term in the M summation. For $M=0$, $M' \wedge M = M'$, and the sum over M' can be recast using the Poisson formula into

$$\frac{1}{\eta\bar{\eta}} \sum_{P \in \mathbb{Z}} (q\bar{q})^{(e_0 + 2P)^2/4g} = \frac{1}{\eta\bar{\eta}} \sum_{P \in \mathbb{Z}} (q\bar{q})^{\Delta_{e_0 + 2P, 0}} \tag{3.16}$$

It gives of course the correct small-q behavior $(q\bar{q})^{-c/24}$, c given by (2.24), (3.6). For $M=\pm 1$, $M' \wedge M=1$ and one gets in the same way

$$\frac{2\cos\pi e_0}{\eta\bar{\eta}} \sum_{P\in\mathbb{Z}} q^{\Delta_{2P,1/2}} \bar{q}^{\bar{\Delta}_{2P,1/2}} \tag{3.17}$$

Recalling that the dimension of the spin S operator (in the c theory) is given by[5]

$$\begin{aligned} x_{H_1} &= \frac{g}{2}\left(m=\frac{1}{2}\right)^2 - \frac{(e_0=g-1)^2}{2g} \\ &= 2\Delta_{0,1/2} + \frac{c-1}{12} \end{aligned} \tag{3.18}$$

we get by (3.17) the correct degeneracy $2\cos\pi e_0 = n$ expected for **S**.

For $M=\pm 2$, $M' \wedge M$ can be either 1 or 2 and the corresponding contribution to Z reads

$$\frac{1}{\eta\bar{\eta}}\left[2\cos\pi e_0 \sum_{P\in\mathbb{Z}} q^{\Delta_{2P,1}} \bar{q}^{\bar{\Delta}_{2P,1}} + (\cos 2\pi_0 - \cos\pi e_0) \sum_{P\in\mathbb{Z}} q^{\Delta_{P,1}} \bar{q}^{\bar{\Delta}_{P,1}}\right] \tag{3.19}$$

while for $M=\pm 3$ one has

$$\begin{aligned} \frac{1}{\eta\bar{\eta}}\bigg[&2\cos\pi e_0 \sum_{P\in\mathbb{Z}} q^{\Delta_{2P,3/2}} \bar{q}^{\bar{\Delta}_{2P,3/2}} \\ &+ \frac{2}{3}(\cos 3\pi e_0 - \cos\pi e_0) \sum_{P\in\mathbb{Z}} q^{\Delta_{2P/3,3/2}} \bar{q}^{\bar{\Delta}_{2P/3,3/2}}\bigg] \end{aligned} \tag{3.20}$$

More generally, each M generates terms as $q^{\Delta_{2P/N,M/2}}$, where $P \wedge N=1$ and N divides M, the prefactors of which can be expressed as polynomials (of degree M) in n, and which depend on the factorization of M into prime integers.

In a Coulomb gas language the charge content is clear: the magnetic charges $m=M/2$ are fixed by the model as multiples of 1/2, and for a given m, one has all possible electric charges e such that the spin em is integer. In (3.5), only $e\in 2\mathbb{Z}$ was observed, as a direct consequence of duality invariance, which is broken here because of the introduction of e_0, (3.9).

One can then write general form for the continuum limit of $\mathscr{Z}_n$,

$$\begin{aligned} \mathscr{Z}_n \to \hat{Z}[g, e_0] = \frac{1}{\eta\bar{\eta}}\bigg\{&\sum_{P\in\mathbb{Z}} (q\bar{q})^{\Delta_{e_0+2P,0}} \\ &+ \sum_{M\in\mathbb{N}^*} \sum_{\substack{P\in\mathbb{Z}, N\in\mathbb{N}^* \\ P\wedge N=1 \\ N/M}} \Lambda(M,N)\, q^{\Delta_{2P/N,M/2}} \bar{q}^{\bar{\Delta}_{2P/N,M/2}}\bigg\} \end{aligned} \tag{3.21}$$

The coefficients $\Lambda(M, N)$ are constructed in the following way. First we decompose M and N into prime integers (recall N divides M)

$$\begin{aligned} M &= p_1^{\alpha_1} \cdots p_k^{\alpha_k} \\ N &= p_1^{\beta_1} \cdots p_k^{\beta_k}, \qquad \beta_1 \leqslant \alpha_1, ..., \beta_k \leqslant \alpha_k \end{aligned} \tag{3.22}$$

and we introduce a contracted expression

$$\langle \cos \pi e_0 p_1^{\gamma_1} \cdots p_k^{\gamma_k} \rangle_c = \sum_{\{0 \leqslant \delta_i \leqslant \inf(\gamma_i, 1)\}} (-1)^{\Sigma \delta_i} \cos \pi e_0 p_1^{\gamma_1 - \delta_1} \cdots p_k^{\gamma_k - \delta_k} \tag{3.23}$$

Then one has

$$\Lambda(M, N) = \sum_{\{\gamma_i : \beta_i \leqslant \gamma_i \leqslant \alpha_i\}} \frac{2}{p_1^{\gamma_1} \cdots p_k^{\gamma_k}} \langle \cos \pi e_0 p_1^{\gamma_1} \cdots p_k^{\gamma_k} \rangle_c \tag{3.24}$$

In the general case when e_0 and g are irrational, $\Delta_{2P/N, M/2} = \Delta_{2P'/N', M'/2}$ only if $(P, N, M) = (\pm P', N', M')$. There are thus no degeneracies which could simplify (3.21).

We now turn to the operator content of (3.21). For c given by (3.6), the degenerate operators[1] have conformal weights obeying the Kac formula

$$h_{r,s} = [(gr - s)^2 - (g-1)^2]/4g \tag{3.25}$$

r, s are nonzero integers of the same sign, while h is obtained as

$$h = \Delta - e_0^2/4g = \Delta + (c-1)/24 \tag{3.26}$$

The first terms in (3.21) involve $\Delta_{e_0 + 2P, 0}$ and for $P < 0$, they correspond by (3.26) to $h_{1, 1-2P}$. This gives exactly the thermal series of Dotsenko and Fateev,[5] i.e., the dimensions of operators generated in the short-distance expansion of the product of several energy operators

$$X_{T_L} = 2h_{1, 1+2L} = -2L + 2L(L+1)/g, \qquad L \geqslant 1 \tag{3.27}$$

One verifies in particular[4]

$$X_{T_1} = 2 - 1/\nu = -2 + 4/g \tag{3.28}$$

For M odd $= 2L + 1$, $P = 0$, the dimensions $\Delta_{0, L+1/2}$ describe in the same way the magnetic series of Refs. 5 and 20,

$$X_{H_L} = \tfrac{1}{8} g(2L+1)^2 - (g-1)^2/2g, \qquad L \geqslant 1 \tag{3.29}$$

These dimensions as well as the other $\Delta_{2P/N,L+1/2}$ correspond to nondegenerate operators for g irrational. For M even $=2L$, $P=0$, the dimensions $\Delta_{0,L}$ correspond to other thermal operators, identified in Ref. 20,

$$\tilde{X}_{T_L}=\tfrac{1}{2}gL^2-(g-1)^2/2g, \qquad L\geqslant 1 \tag{3.30}$$

which are also nondegenerate. The $\Delta_{2P/N,L}$ are also nondegenerate, except for $N=1$, $P<0$. In this case $\Delta_{-2P,L}$ corresponds using (3.26) to $h_{L,-2P}$.

The characters of the corresponding Virasoro algebra are defined by

$$\chi_h=q^{h-c/24}\sum D_K q^K \tag{3.31}$$

where D_K is the number of independent secondary fields at level K. If $h\neq h_{r,s}$, χ_h reads simply

$$\chi_h=\frac{q^{h-c/24}}{\prod_{N=1}^{\infty}(1-q^N)}=\frac{q^{\Delta}}{\eta(q)} \tag{3.32}$$

while[11]

$$\chi_{h_{r,s}}=\frac{q^{h_{r,s}-c/24}-q^{h_{r,-s}-c/24}}{\prod_{N=1}^{\infty}(1-q^N)} \tag{3.33}$$

It is thus clear that (3.21) can be written as a quadratic form

$$\hat{Z}[g,e_0]=\sum_{h,\bar{h}} R_{h,\bar{h}}\chi_h\chi_{\bar{h}}^* \tag{3.34}$$

Except for $n=2$, 1, 0, the $R_{h\bar{h}}$ can be noninteger. In particular, the factor R corresponding to the spin operator is simply n, which becomes negative for $n<0$. In fact, as soon as $n<2$, some of the R become negative, for instance, the prefactor associated with $\Delta_{2P/3,3/2}$, which is

$$\tfrac{2}{3}(\cos 3\pi e_0-\cos \pi e_0)=n(n-2)(n+2)/3$$

In the standard statistical mechanics models, the $R_{h\bar{h}}$ must be positive integers: this is because $\mathscr{Z}$ is obtained as the trace of the power of a certain transfer matrix. The expression (3.34) shows that there is no transfer matrix for the $O(n)$, $n\in\mathbb{R}$ model whose trace allows one to close the cylinder into a torus. Indeed, the matrices introduced so far[20] work in a nonsymmetric way, a special role being played by the "left part" of the strip where some connectivities between bonds of (3.2) are defined.

Finally, we want to discuss the dependence on n of the R in (3.34). We consider for simplicity the spinless primary operators only. Then, the first terms read

$$n(q\bar{q})^{\Delta_{0,1/2}} + \frac{n^2+n-1}{2}(q\bar{q})^{\Delta_{0,1}} + \frac{n^3-n}{3}(q\bar{q})^{\Delta_{0,3/2}}$$
$$+\left(\frac{n^4}{4}-\frac{3n^2}{4}+\frac{n}{2}\right)(q\bar{q})^{\Delta_{0,2}} + \cdots \tag{3.35}$$

It is very tempting to identify the polynomials in (3.35) with dimensions of the irreducible representations of the $O(n)$ group, but this works only for the first (vector representation S^{μ}) and the second term (symmetric traceless tensor $S^{\mu\nu}$). The other terms are not these dimensions, nor combinations with positive integer coefficients. This in fact is not surprising, since (3.35) is valid for three integer values of n only ($n=0, 1, 2$). We cannot expect that the analytic continuation fixes all the polynomials, but only those of degree one and two. We notice that in the low-temperature phase of (3.2), which is also critical, renormalizing onto (2.1) with g given by another branch of (3.3), the prefactor of the spin operator in (3.35) is still n. This shows the limits of the analogy with Goldstone modes (see Ref. 21 and references therein), in which one would expect instead $n-1$.

5. When g is rational, many cancellations can simplify the expression (3.21), as is observed for $n=0, 1, 2$. It seems, however, quite difficult to write down explicit expressions. Only for $n=-1, -2$ have we obtained simple results,

$$\begin{aligned}\mathscr{Z}_{-1} &\to (Z_c[5/3, 3/2] - Z_c[5/3, 1/2])/2 \\ \mathscr{Z}_{-2} &\to Z_c[2, 1] - Z_c[2, 1/2] = 0\end{aligned} \tag{3.36}$$

This last result deserves some comment.

To obtain a nontrivial result for $\mathscr{Z}_{-2}$, we have to take the derivative of $\hat{Z}[g, e_0]$ with respect to n at the point $n=-2$. The only nonvanishing term is

$$\frac{\partial g}{\partial n}\frac{\partial}{\partial g}(Z_c[g, 1] - Z_c[g, 1/2])_{g=2} \tag{3.37}$$

i.e., up to a multiplicative constant

$$\frac{\tau_I}{\eta\bar{\eta}}\sum_{e,m\in\mathbb{Z}}\left(\frac{e^2}{8}-\frac{m^2}{2}\right)(q\bar{q})^{e^2/8+m^2/2}\left(\frac{q}{\bar{q}}\right)^{em/2} \tag{3.38}$$

Using the Jacobi identity

$$\frac{1}{2}\sum_{P\in\mathbb{Z}}(-1)^P(2P+1)\,q^{(P^2+P)/2}=\left[\prod_{N=1}^{\infty}(1-q^N)\right]^3 \tag{3.39}$$

one gets finally

$$\left.\frac{\partial}{\partial n}\mathscr{Z}_n\right|_{n=-2}\to\tau_I\eta^2\bar{\eta}^2 \tag{3.40}$$

The normalization is chosen such that the identity operator is non-degenerate. Equation (3.40) is the renormalized value of det Δ on the torus. This result agrees with arguments of Parisi and Sourlas[22] that $n=-2$ is a free fermionic theory with action

$$\mathscr{A}=\int\bar{\psi}\,\Delta\psi\,dS \tag{3.41}$$

where ψ ($\bar{\psi}$) are two independent Grassmannian fields. Equation (3.40) can also be related to the counting of Hamiltonian graphs on the Manhattan lattice.[23]

We note also that for $g=p/p'$, $c=1-6(p-p')^2/pp'$, (3.21) does not give any of the minimal partition functions for unitary models (except for $n=0, 1$). This is easily seen, since in the expansion (3.34), the identity has a prefactor one, while the spin has the prefactor $n=-2\cos(p/p')\notin\mathbb{N}$.

6. A quite interesting physical case is the limit $n\to 0$, which is known to describe polymers.[24] Here, following (3.3), $g=3/2$. The object $\hat{Z}[3/2, e]$ is then the continuum limit of the generating function

$$G=\sum_{\text{graphs}}(2\cos\pi e)^{\mathscr{N}_P}\,\mu^{-\mathscr{N}_B} \tag{3.42}$$

the graphs being formed by $\mathscr{N}_P$ polygons nonhomotopic to a point and $\mu=T_c$ ($n=0$) is the connectivity constant for self-avoiding walks on the lattice. In a similar way, the derivative $(\partial/\partial n)\,\hat{Z}(g, e_0)$ is the limit of the generating function

$$G=\sum_{\text{graphs}}\mu^{-\mathscr{N}_B} \tag{3.43}$$

the sum being taken over all configurations with one single polygon on the torus [$(\partial g/\partial n)\,\partial\hat{Z}/\partial g$ selects only those where the polygon is homotopic to a point, and $(\partial e_0/\partial n)\,\partial\hat{Z}/\partial e_0$ those where it is not].

4. PARTITION FUNCTIONS OF CRITICAL Q-STATE POTTS MODELS ($Q \in \mathbb{R}$) ON A TORUS

1. We turn now to the same study for the Potts model, defined initially for Q integer on, say, the square lattice $\mathscr{L}$ by the action[25]

$$\mathscr{A} = -\frac{1}{T} \sum_{\langle jk \rangle} \delta_{\sigma_j \sigma_k} \tag{4.1}$$

($\sigma = 1, ..., Q$). The analytic continuation to $Q \in \mathbb{R}$ is obtained using the high-temperature expansion

$$\mathscr{Z}_Q = \sum_{\text{graphs}} (e^{1/T} - 1)^{\mathscr{N}_B} Q^{\mathscr{N}_C} \tag{4.2}$$

where the graphs are obtained by putting $\mathscr{N}_B$ bonds on the edges of the lattice, which form $\mathscr{N}_C$ clusters, i.e., connected components (including isolated points). Equation (4.2) is more easily handled using a polygon decomposition of the surrounding lattice $\mathscr{S}$,[25] here another square lattice (see Fig. 4). If $\mathscr{N}_L$ is the number of loops in a given graph of (4.2) and $\mathscr{N}_S$ the total number of sites in $\mathscr{L}$, then by Euler's relation

$$\mathscr{N}_L = \mathscr{N}_B + \mathscr{N}_C - \mathscr{N}_S \tag{4.3}$$

In a plane the number $\mathscr{N}_p$ of polygons on $\mathscr{S}$ reads

$$\mathscr{N}_p = \mathscr{N}_L + \mathscr{N}_C \tag{4.4}$$

Hence (4.2) can be rewritten as

$$\mathscr{Z}_Q = Q^{\mathscr{N}_S/2} \sum_{\text{graphs}} [(e^{1/T} - 1)\, Q^{-1/2}]^{\mathscr{N}_B} Q^{\mathscr{N}_P/2} \tag{4.5}$$

Model (4.5) is known to have a second-order phase transition for $Q \in [0, 4]$, the critical temperature being such that $(e^{1/T} - 1)\, Q^{-1/2} = 1$. One then transforms (4.5) in the same way as for the $O(n)$ model by considering a polygon arbitrarily oriented as a wall between two regions of constant height differing by $\pm\varphi_0$. With a factor e^{iu} (e^{-iu}) for each left (right) turn one has, in the plane $\mathscr{Z}_Q = \mathscr{Z}_{\text{SOS}}$ if $Q^{1/2} = 2 \cos 4u$. The choice[4] $\varphi_0 = \pi/2$ leads to the renormalized coupling g such that

$$Q = 2 + 2\cos(\pi g/2), \qquad g \in [2, 4] \tag{4.6}$$

2. As before, $Z_c[g/4, 1]$ is the continuum limit of a modified model $\tilde{\mathscr{Z}}_Q$ defined as in (4.5) but with a factor 2 for each polygon nonhomotopic

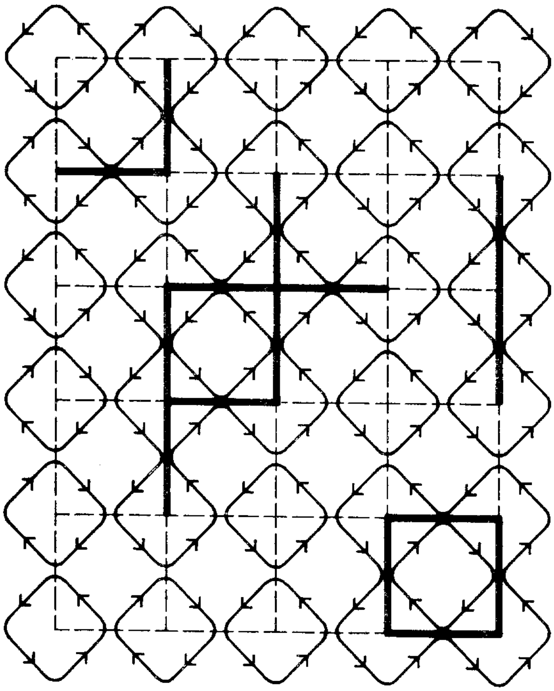


Fig. 4. A typical graph in the high-temperature expansion of $\mathscr{Z}_Q$, (4.2), and its alternative polygon representation. After arbitrary orientation, the polygons are considered as walls between regions of constant height in a solid-on-solid model.

to a point.[11] On a strip, this is easily repaired by putting charges $\pm e_0$ at infinity such that $Q^{1/2} = 2\cos[(\pi/2)\, e_0]$, resulting in

$$\begin{aligned} e_0 &= \pm(2 - g/2) \bmod 4 \\ c &= 1 - 6(2 - g/2)^2/g \end{aligned} \tag{4.7}$$

To give the correct weights to all polygons on the torus one has to consider the same object $\hat{Z}[g, e_0]$, (3.9), as for the $O(n)$ model, with e_0, g defined by (4.6), (4.7). This, however, does not yet describe the original Potts model (4.2). Indeed, on the torus, although (4.3) remains valid, (4.4) can be violated when a cluster has a "cross topology," i.e., winds around (at least) two independent noncontractible cycles of the torus (Fig. 5), in which case $\mathscr{N}_L + \mathscr{N}_C - \mathscr{N}_P = 2$. Equation (4.5) gives to such graphs the relative weight 1, while it should be Q. We thus have to add to $\hat{Z}[g, e_0]$ a factor $(Q-1)$ times the partition function restricted to clusters with cross topology. These are easily selected by giving a weight 0 to each polygon nonhomotopic to a point, i.e., by adding a charge $e_0' = 1$. The remaining configurations contain

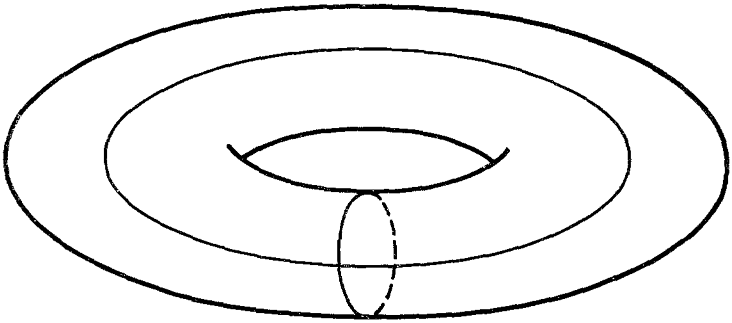


Fig. 5. Schematic representation of a cluster with "cross topology."

a cross cluster of either occupied or empty bonds. Since these two classes are in a one-to-one correspondence by duality, we avoid double counting by a factor 1/2. We get finally

$$\mathscr{Z}_Q \to \hat{Z}[g, e_0] + \tfrac{1}{2}(Q-1)(Z_c[g, 1] - Z_c[g, 1/2]) \tag{4.8}$$

One recovers from (4.8) the correct results for $Q = 1, 2, 3$ (see the Appendix). For $Q = 4$ it gives

$$\mathscr{Z}_4 \to (3Z_c[4, 1] - Z_c[4, 1/2])/2 \tag{4.9}$$

in agreement with the work of Yang.[12]

3. The same general comments are valid here as for the $O(n)$ model, and we simply make explicit the operator content of (4.8). The Kac formula reads

$$h_{rs} = [(4r - gs)^2 - (4-g)^2]/16g \tag{4.10}$$

while

$$h = \Delta + (c-1)/24 \tag{4.11}$$

The first term of $\hat{Z}$ involves $\Delta_{e_0+2P,0}$. For $P \geqslant 0$, it corresponds to using (4.11) to $h_{P+1,1}$, i.e., the thermal series of Dotensko and Fateev,[5]

$$X_{T_L} = 2h_{L+1,1} = -L + 2L(L+2)/g, \qquad L \geqslant 1 \tag{4.12}$$

One verifies in particular[4]

$$X_{T_1} = -1 + 6/g \tag{4.13}$$

The spinless primary operators labeled by $\Delta_{0,M/2}$ describe the "hull dimensions" recently introduced in Ref. 26. The additional term in (4.8) contains in particular in the zero magnetic charge ($m=0$) sector the operators with $\Delta_{2L-1,0}$, which correspond to the magnetic series of Dotsenko and Fateev,

$$X_{H_L}=\frac{(2L-1)^2}{2g}-\frac{(2-g/2)^2}{2g} \tag{4.14}$$

The prefactors are all equal to $Q-1$, in agreement with the symmetry of the model.

4. Our construction is also valid for the tricritical Potts model, known[27] to renormalize onto (2.1) with g given by another branch of (4.7), $g\in[4, 6]$. For the case $Q=3$ ($e_0=\frac{1}{3}$) one finds from (4.8)

$$\mathscr{Z}_3\rightarrow\frac{1}{2}\left(Z_c\left[\frac{g}{4},2\right]-Z_c\left[\frac{g}{4},1\right]-Z_c\left[\frac{g}{4},3\right]+Z_c\left[\frac{g}{4},6\right]\right) \tag{4.15}$$

The critical model corresponds to $g=10/3$, the tricritical one to $g=14/3$. Hence

$$\mathscr{Z}_3(\text{critical})\rightarrow\frac{1}{2}\left(Z_c\left[\frac{10}{3},1\right]-Z_c\left[\frac{10}{3},\frac{1}{2}\right]-Z_c\left[\frac{10}{3},\frac{3}{2}\right]+Z_c\left[\frac{10}{3},3\right]\right) \tag{4.16}$$

while

$$\mathscr{Z}_3(\text{tricritical})\rightarrow\frac{1}{2}\left(Z_c\left[\frac{14}{3},1\right]-Z_c\left[\frac{14}{3},\frac{1}{2}\right]-Z_c\left[\frac{14}{3},\frac{3}{2}\right]+Z_c\left[\frac{14}{3},3\right]\right) \tag{4.17}$$

which are easily seen to coincide with the corresponding expressions for the unitary minimal conformal theories (see the Appendix).

5. CONCLUSION

We have thus shown how the Coulomb gas mapping of the $O(n)$ and Q-state Potts models allows an explicit calculation of their partition function on the torus. This procedure is an alternative to the systematic search for modular-invariant partition functions. Formulas collected in the Appendix suggest that a similar construction is possible for other models, such as those classified in Ref. 10 and constructed in Ref. 28. We hope that this procedure can be extended to the derivation of the correlation functions on the torus, as well as the calculation of partition functions on surfaces of higher genus.

An intriguing feature is that in all conformal theories considered so far, the partition functions may be expressed as linear combinations of functional integrals over free boson fields, with various types of boundary conditions. This extends to models with $c>1$, where several boson fields are necessary. We intend to return to these questions in a later publication.

APPENDIX

In this Appendix, we show how all the partition functions of minimal conformal theories classified in Ref. 10 may be reexpressed in terms of Coulombic partition functions.

We recall that minimal conformal theories have a central charge

$$c = 1 - 6(p - p')^2/pp' \tag{A.1}$$

with p, p' two coprime integers, and involve a finite number of primary fields, of conformal dimensions h and $\bar{h}$ given by Kac's formula:

$$h_{rs} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \tag{A.2}$$

with the constraint

$$1 \leqslant r \leqslant p' - 1, \qquad 1 \leqslant s \leqslant p - 1 \tag{A.3}$$

We also recall that the most general modular invariant of the form (1.2) constructed with the conformal characters pertaining to representations (A.2)–(A.3) is a linear combination of contributions associated with the various possible factorizations of p and p'.[10,19] It turns out that in all the invariants with positive coefficients $N_{h\bar{h}}$ in (1.2), one of these two numbers, say p, factorizes trivially as $p = 1 \times p$, while the variety of invariants comes from factorizations of the other as $p' = p'_1 p'_2$. We denote the corresponding invariant

$$Z^{p'_1 \times p'_2}$$

and refer to Ref. 10 for its explicit construction. In Ref. 11, it was also proved that each such term may be written as a difference of two Coulombic partition functions defined in (2.16):

$$Z^{(p'_1 \times p'_2)} = \frac{1}{2}\left[Z_c\left(\frac{pp'_2}{p'_1}, 1\right) - Z_c\left(\frac{pp'_1}{p'_2}, 1\right)\right] \tag{A.4}$$

It is then an easy matter to transcribe all the physical invariants listed in Ref. 10 in terms of Coulombic functions:

For any p, p'

$$Z = Z^{(1\times p')} = \frac{1}{2}\left[Z_c(pp', 1) - Z_c\left(\frac{p}{p'}, 1\right)\right]$$
$$= \frac{1}{2}\left[Z_c(pp', 1) - Z_c\left(pp', \frac{1}{p'}\right)\right]$$
$$= \frac{1}{2}\left[Z_c\left(\frac{p}{p'}, p'\right) - Z_c\left(\frac{p}{p'}, 1\right)\right] \tag{A.5}$$

For $p' = 0$, mod 2,

$$Z = Z^{(1\times p')} + Z^{(p'/2\times 2)}$$
$$= \frac{1}{2}\left[Z_c\left(\frac{4p}{p'}, 1\right) - Z_c\left(\frac{4p}{p'}, \frac{1}{2}\right) - Z_c\left(\frac{4p}{p'}, \frac{1}{p}\right) + Z_c\left(\frac{4p}{p'}, \frac{1}{2p}\right)\right] \tag{A.6}$$

For $p' = 12$

$$Z = Z^{(1\times 12)} + Z^{(6\times 2)} + Z^{(4\times 3)}$$
$$= \frac{1}{2}\left[Z_c(g, 1) - Z_c\left(g, \frac{1}{2}\right) - Z_c\left(g, \frac{1}{3}\right) + Z_c\left(g, \frac{1}{4}\right)\right.$$
$$\left. + Z_c\left(g, \frac{1}{6}\right) - Z_c\left(g, \frac{1}{12}\right)\right] \tag{A.7}$$

For $p' = 18$

$$Z = Z^{(1\times 18)} + Z^{(9\times 2)} + Z^{(6\times 3)}$$
$$= \frac{1}{2}\left[Z_c(g, 1) - Z_c\left(g, \frac{1}{2}\right) - Z_c\left(g, \frac{1}{3}\right) + Z_c\left(g, \frac{1}{6}\right)\right.$$
$$\left. + Z_c\left(g, \frac{1}{9}\right) - Z_c\left(g, \frac{1}{18}\right)\right] \tag{A.8}$$

For $p' = 30$

$$Z = Z^{(1\times 30)} + Z^{(15\times 2)} + Z^{(10\times 3)} + Z^{(6\times 5)}$$
$$= \frac{1}{2}\left[Z_c(g, 1) - Z_c\left(g, \frac{1}{2}\right) - Z_c\left(g, \frac{1}{3}\right) - Z_c\left(g, \frac{1}{5}\right) + Z_c\left(g, \frac{1}{6}\right)\right.$$
$$\left. + Z_c\left(g, \frac{1}{10}\right) + Z_c\left(g, \frac{1}{15}\right) - Z_c\left(g, \frac{1}{30}\right)\right] \tag{A.9}$$

In the three latter cases, g denotes $g = pp'$.

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